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Numerical computations to pulse splitting phenomena in the thermal waves through an absorbing medium☆

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Abstract

Nonlinear diffusion with strong absorption causes "total extinction in finite time" and "finite propagation of the initial support". As the dynamical behavior of the support of pulses there is a possibility of the support to split, when the initial function has two local maxima. Thus *pulse splitting phenomena* appear. In this paper the difference scheme which predicts such phenomena is proposed. Moreover, the criterion for *pulse nonsplitting phenomena* is also given. Several numerical experiments are presented.

Keywords: Difference scheme; Nonlinear diffusion; Interfaces; Finite extinction; Support splitting

1. Introduction

Nonlinear diffusion equations exhibit a wide variety of wave phenomena in the several fields of fluid dynamics, plasma physics and population dynamics. A commonly used form which describes such phenomena is the degenerate parabolic equation with an additional lower order term. In plasma physics the transport of the thermal energy through an absorbing medium is governed by the interaction between diffusion and absorption. As remarkable properties there appear "total extinction in finite time" and "finite propagation of the initial support", which are caused by the strong absorption and the degeneracy of diffusion rate, respectively. From these properties we may expect the appearance of the interesting behavior of thermal waves. The support of the pulses in the thermal

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waves becomes disconnected in the case where absorption can cool the medium faster than diffusion supplies heat from the hot area; that is, the pulse splitting occurs. In the opposite case, the pulse never splits.

To describe such phenomena, we use the degenerate parabolic equation with absorption, where the thermal conductivities and sinks are assumed to be temperature dependent. This equation is also well-known as the description of the flow of the liquids through the homogeneous porous medium [1]. In the following we are concerned with the one-dimensional initial value problem:

$$v_t = (v^m)_{xx} - cv^p, \quad x \in \mathbb{R}^1, \quad t > 0, \tag{1}$$

$$v(0,x) = v^0(x), \quad x \in \mathbb{R}^1, \tag{2}$$

where m, p and $v^0(x)$ satisfy the following conditions:

(A-1) $m(>1)$, $p(>0)$, and $c(\geqslant 0)$ are constants and $m+p \geqslant 2$;
(A-2) $v^0(x) \in C^0(\mathbb{R}^1)$ is nonnegative and has compact support.

In a heated plasma v denotes the temperature and $-cv^p$ describes the losses caused by radiation. We may take $p = 0.5$ for bremsstrahlung radiation and $0.5 \leqslant p \leqslant 2$ for synchrotron radiation [11]. The existence and uniqueness of a weak solution and the property of the finite propagation of the support are established by many authors [5–7,10]. The dynamical behavior of the support is classified into two cases.

(B-1) For $c = 0$, or $c > 0$ and $p \geqslant 1$ $\operatorname{supp} v(t,\cdot)$ monotonously expands as t increases; that is, $\operatorname{supp} v(t,\cdot)$ never becomes disconnected for $t > 0$, when $\operatorname{supp} v^0(x)$ is connected. Thus the *pulse splitting phenomena* never appear;
(B-2) for $c > 0$ and $0 < p < 1$ $\operatorname{supp} v(t,\cdot)$ expands and/or shrinks and $v(t,x)$ vanishes identically at some finite time $T^* > 0$; that is, $v(t,x) \equiv 0$ for $t \geqslant T^*$ and $v(t,x) \not\equiv 0$ for $t < T^*$. Such an absorption is said to be a strong absorption.

In Case (B-2) there is a possibility of the *pulse splitting phenomena*, when $v^0(x)$ has two local maxima (see Fig. 3). Rosenau and Kamin [11] suggested this possibility by numerical computations. But the mathematical justification was not discussed. This motivates us to investigate such phenomena in both numerical and analytical points of view. For this end the difference scheme with criteria which predict the following behavior of pulses is needed:

(C-1) The *pulse splitting phenomena* appear for the initially connected pulses, and thereafter theses pulses evolve separately until one of them vanishes;
(C-2) The *pulse splitting* phenomena never appear for $t > 0$.

In this paper we propose such criteria in the specific case where

$$m + p = 2 \quad \text{and} \quad 0 < p < 1, \tag{3}$$

and demonstrate some numerical examples. Our mathematical methods are based on finite difference methods, the comparison theorem [2] and Kersner's explicit solution [8]. Unfortunately, in the case where $m+p \neq 2$, $m > 1$ and $0 < p < 1$, we are unable to succeed in constructing the finite difference scheme with convergence and to find any explicit solution. In the following we assume (3).

2. Criteria for (C-1) and (C-2)

2.1. Support splitting phenomena

To state the criterion for (C-1) we introduce the set $\mathbf{W}$ consisting of all nonnegative functions $\varphi(x) \in C^0(\mathbb{R}^1)$ with compact support which satisfy the following conditions:

(W-1) $\Phi_x(x) \in L^\infty(\mathbb{R}^1) \cap BV(\mathbb{R}^1)$, where $\Phi(x) = \varphi(x)^{m-1}$;
(W-2) $\Phi_x(x)$ is absolutely continuous on $\mathbf{I} \equiv [\ell(\Phi),\ r(\Phi)]$ and $\mathrm{ess.inf}_{\mathbf{I}}\, \Phi_{xx}(x)$ is finite, where $\ell(\Phi)$ and $r(\Phi)$ are the left and right interfaces defined by

$$\begin{cases} \ell(\Phi) = \sup\{\xi \mid \Phi(x) = 0 \text{ on } (-\infty, \xi]\}, \\ r(\Phi) = \inf\{\xi \mid \Phi(x) = 0 \text{ on } [\xi, \infty)\}. \end{cases} \tag{4}$$

We put

$$c' = (m-1)c, \quad a = \frac{m}{m-1}, \quad u^0(x) = (v^0(x))^{m-1},$$

$$C_0 = \|u^0\|_\infty, \quad C_1 = \|u_x^0\|_\infty \text{ and } C_2 = -\mathrm{ess.inf}_{\mathbf{I}}\, u_{xx}^0(x),$$

where $\|\cdot\|_\infty$ denotes $\|\cdot\|_{L^\infty(\mathbb{R}^1)}$. Then we have

Theorem 1. *For $\alpha_1 < \beta_1 < \gamma_1 < \gamma_2 < \beta_2 < \alpha_2$ let $v^0(x) \in \mathbf{W}$ satisfy*

$$v^0(x) > 0 \text{ on } (\alpha_1,\ \alpha_2) \quad \text{and} \quad \mathrm{supp}\, v^0(x) = [\alpha_1,\ \alpha_2]. \tag{5}$$

Assume

$$\frac{u^0(\beta_j)}{c' + mC_0C_2} > \frac{\|u^0\|_{L^1(\gamma_1,\gamma_2)}}{c'(\gamma_2 - \gamma_1) - (m+a)C_0 TV(u_x^0)} > 0 \quad (j = 1, 2). \tag{6}$$

Then there exist $\tilde{t} > 0$ and $\tilde{x} \in [\gamma_1,\ \gamma_2]$ such that $v(\tilde{t}, \tilde{x}) = 0$ and $v(\tilde{t}, \beta_j) > 0$ $(j = 1, 2)$.

Also, if $u^0(x)$ satisfies

$$a\|u_x^0\|_\infty^2 < c', \tag{7}$$

then $v(t, \tilde{x}) = 0$ for $t \geqslant \tilde{t}$ and there exists a positive constant t^* such that $\mathrm{supp}\, v(t,\cdot)$ is disconnected for each $t \in (\tilde{t},\ \tilde{t} + t^*)$.

2.2. Support nonsplitting phenomena

We state the criterion for (C-2) by using Kersner's explicit solution [9], which is obtained in the case where $m + p = 2$ and $m > 1$. For given constants $\rho > 0$ and $\sigma > 0$ the solution is defined by $v(t,x) = K(t,x,\rho,\sigma)$ where K is given by

$$K(t,x,\rho,\sigma) = (b_1 t + b_2(\sigma))^{-\frac{1}{m-1}}$$
$$\times [a_1(\rho,\sigma)(b_1 t + b_2(\sigma))^{\frac{2}{m+1}} - a_2(b_1 t + b_2(\sigma))^2 - x^2]_+^{\frac{1}{m-1}}. \tag{8}$$

Here $[g]_+ = \max\{g, 0\}$,

$$a_1(\rho,\sigma) = \frac{c(m-1)^4\sigma^2 + 4m^2\rho^2}{4m^2\{(m-1)\sigma\}^{\frac{2}{m+1}}}, \quad a_2 = \frac{c(m-1)^2}{4m^2}, \tag{9}$$

$$b_1 = \frac{2m(m+1)}{m-1}, \quad b_2(\sigma) = (m-1)\sigma. \tag{10}$$

This solution satisfies (1)–(2) with $v^0(x) = K(0,x,\rho,\sigma)$ in the weak sense and becomes a classical solution in the open set wherein $K(t,x,\rho,\sigma) > 0$. It identically vanishes at the extinction time $t = T^*(\rho,\sigma)$ given by

$$T^*(\rho,\sigma) = \frac{1}{b_1}\left[\left\{\frac{a_1(\rho,\sigma)}{a_2}\right\}^{\frac{m+1}{2m}} - b_2(\sigma)\right]. \tag{11}$$

Putting

$$f(t,\rho,\sigma) = a_1(\rho,\sigma)\{b_2(\sigma)^{-\frac{m-1}{m+1}} - (b_1 t + b_2(\sigma))^{-\frac{m-1}{m+1}}\} - 2ma_2 t, \tag{12}$$

we find that $f(t,\rho,\sigma) > 0$ holds on $(0, T^*(\rho,\sigma))$, and obtain

Theorem 2. *Let $v^0(x)$ satisfy Condition (A-2) and be an even function; that is, $v^0(x) = v^0(-x)$ holds on $\mathbb{R}^1$. Assume $v^0(x)$ has just two points of local maximum and there exist positive constants $\sigma_i (i = 1,2)(\sigma_1 < \sigma_2)$ such that*

$$K(0,x,\rho,\sigma_2) \leqslant v^0(x) \leqslant K(0,x,\rho,\sigma_1), \tag{13}$$

$$\frac{\rho^2}{m-1}\left(\frac{1}{\sigma_1} - \frac{1}{\sigma_2}\right) < f(T^*(\rho,\sigma_2),\rho,\sigma_1), \tag{14}$$

where $\operatorname{supp} v^0 = [-\rho, \rho]$. *Then* $\operatorname{supp} v(t,\cdot)$ *never becomes disconnected, where v denotes the solution of* (1)–(2).

3. Difference scheme

We put $u = v^{m-1}$ and rewrite (1)–(2) as follows:

$$u_t = muu_{xx} + a(u_x)^2 - c', \tag{15}$$

$$u(0,x) = u^0(x) \equiv (v^0(x))^{m-1}, \tag{16}$$

where the term of strong absorption is written as the constant $-c'$ by the assumption $m + p = 2$. Our difference scheme approximates the problem (15)–(16) instead of (1)–(2), which is stated as follows:

Find the sequence $\{u_h^n\}_{n=1,2,\ldots} \subset V_h$ for each $u_h^0 \in V_h$ such that

$$u_h^{n+1} = S_{h,k}u_h^n \equiv (P_{h,\kappa})^\mu \left(\prod_{j=1}^{\nu} H_{h,\tau_j}\right) D_{h,k}u_h^n \quad \text{for } n = 0,1,2,\ldots, \tag{17}$$

where $t_0 = 0$, $\ell(u_h^0) = \ell(u^0)$, $r(u_h^0) = r(u^0)$ and $u_h^0(x_i) = u^0(x_i)$ for all $i \in \mathbb{Z}$. Here $D_{h,k}$, H_{h,τ_j} and $P_{h,\kappa}$ are difference operators approximating $u_t = Du = -c'$, $u_t = Hu = a(u_x)^2$ and $u_t = Pu = muu_{xx}$, respectively, and h is a space mesh width. The variable time steps $k \equiv k_{n+1} \equiv t_{n+1} - t_n$, $\tau_j \equiv \tau_{n+1,j}$ and $\kappa \equiv \kappa_{n+1}$ are determined in the description of $D_{h,k}$, H_{h,τ_j} and $P_{h,\kappa}$, respectively. The positive integers $\mu \equiv \mu_{n+1}$ and $v \equiv v_{n+1}$ satisfy

$$\sum_{j=1}^{v_{n+1}} \tau_{n+1,j} = \mu_{n+1}\kappa_{n+1} = k_{n+1}, \tag{18}$$

and V_h is the set of the nonnegative continuous functions $u_h = u_h(x)$ with the following properties:

(i) u_h has compact support;
(ii) u_h is linear on each interval $[x_i, x_{i+1}]$ $(i \in \mathbb{Z})$, where

$$x_i = \begin{cases} ih & \text{for } i \in \mathbb{Z} \setminus \{L-1, R+1\}, \\ \ell & \text{for } i = L-1, \\ r & \text{for } i = R+1, \end{cases} \tag{19}$$

$$L = L(\ell) \equiv \min\{i \in \mathbb{Z} \mid ih > \ell\}, \quad \ell = \ell(u_h), \tag{20}$$

$$R = R(r) \equiv \max\{i \in \mathbb{Z} \mid ih < r\}, \quad r = r(u_h). \tag{21}$$

The left and right numerical interfaces are defined by

$$\ell_n = \ell(u_h^n) \quad \text{and} \quad r_n = r(u_h^n) \quad \text{for} \quad n = 0, 1, 2, \ldots, \tag{22}$$

respectively. When $D_{h,k}u_h^{n^*} \equiv 0$ holds for some integer $n^* > 0$, we define the numerical extinction time by $T_h^* = t_{n^*+1} \equiv t_{n^*} + k_{n^*+1}$, and stop the numerical computation. We put

$$h_i = x_{i+1} - x_i, \quad u_i = u_h(x_i),$$

$$\delta u_i = (u_{i+1} - u_i)/h_i, \quad \delta^2 u_i = 2(\delta u_i - \delta u_{i-1})/(h_i + h_{i-1}).$$

Difference operator $D_{h,k}$: Putting $u_h' = D_{h,k}u_h$, we define it by

$$\ell(u_h') = \max\{\ell(u_h),\ (L'-1)h\}, \quad r(u_h') = \min\{r(u_h),\ (R'+1)h\}, \tag{23}$$

$$u_h'(x_i) = u(k, x_i) \quad \text{for all } i \in \mathbb{Z}, \tag{24}$$

where $L' = L(\ell(u(k,\cdot)))$, $R' = R(r(u(k,\cdot)))$, and

$$u(k,x) = \max\{u_h(x) - c'k,\ 0\} \quad \text{for } x \in \mathbb{R}^1. \tag{25}$$

The time step k is determined by

$$k = \frac{1}{c'}\max(u_L, u_{L+1}) \text{ for the approximation to the left interface,} \tag{26}$$

or

$$k = \frac{1}{c'}\max(u_R, u_{R-1}) \text{ for the approximation to the right interface.} \tag{27}$$

Difference operator $H_{h,\tau}$:

$$\ell(H_{h,\tau}u_h) = \ell(u_h) - a\delta u_{L-1}\tau, \quad r(H_{h,\tau}u_h) = r(u_h) - a\delta u_R\tau, \tag{28}$$

$$H_{h,\tau}u_h(x_i) = \begin{cases} u_i + a(\delta u_i)^2\tau & \text{if } i \in S^+ = S_S^+ \cup S_R^+, \\ u_i + a(\delta u_{i-1})^2\tau & \text{if } i \in S^- = S_S^- \cup S_R^-, \\ u_i & \text{if } i \in S^0, \\ (L'h - \ell')\delta u_{L-1} & \text{if } i = L' = L-1, \\ (R'h - r')\delta u_R & \text{if } i = R' = R+1, \\ 0 & \text{if } i \in \mathbb{Z} \setminus \{L', \ldots, R'\}, \end{cases} \tag{29}$$

where

$$\ell' = \ell(H_{h,\tau}u_h), \quad r' = r(H_{h,\tau}u_h), \quad L' = L(\ell'), \quad R' = R(r'),$$

$$S_S^+ = \{i \in \{L, \ldots, R\} \mid \delta u_{i-1} < \delta u_i \quad \text{and} \quad \delta u_{i-1} > -\delta u_i\},$$

$$S_S^- = \{i \in \{L, \ldots, R\} \mid \delta u_{i-1} < \delta u_i \quad \text{and} \quad \delta u_{i-1} \leqslant -\delta u_i\},$$

$$S_R^+ = \{i \in \{L, \ldots, R\} \mid \delta u_{i-1} \geqslant \delta u_i > 0\},$$

$$S_R^- = \{i \in \{L, \ldots, R\} \mid 0 > \delta u_{i-1} \geqslant \delta u_i\},$$

$$S^0 = \{i \in \{L, \ldots, R\} \mid \delta u_{i-1} \geqslant 0 \geqslant \delta u_i\}.$$

The time step τ satisfies

$$a\|(u_h)_x\|_\infty\tau = \min\left\{\frac{h}{4}, Lh - \ell(u_h), r(u_h) - Rh\right\}. \tag{30}$$

We note that $H_{h,\tau}u_h(x_i)$ coincides with the exact solution $u(\tau, x_i)$ of $u_t = Hu$ with the initial value $u(0,x) = u_h(x)$. The numerical left and right interfaces (28) are well-known as the Rankine-Hugoniot jump equation determining the lines on which the shock occurs in $w_t = a(w^2)_x$ derived from $u_t = Hu$ by putting $w = u_x$.

Difference operator $P_{h,\kappa}$:

$$\ell(P_{h,\kappa}u_h) = \ell(u_h), \quad r(P_{h,\kappa}u_h) = r(u_h), \tag{31}$$

$$P_{h,\kappa}u_h(x_i) = u_i + \kappa m u_i \delta^2 u_i \quad \text{for all } i \in \mathbb{Z}. \tag{32}$$

The time step κ is the largest number satisfying

$$m\|u_h\|_\infty\kappa\left\{\frac{1}{h^2} + \frac{2}{h(h+h_j)}\right\} \leqslant 1 \quad \text{for } j = L-1 \text{ and } R, \tag{33}$$

$$\frac{4m\|(u_h)_x\|_\infty\kappa}{h_{j-1} + h_j} \leqslant 1 \quad \text{for } j = L \text{ and } R. \tag{34}$$

We need the basic estimates for $u_h^{n+1} = S_{h,k} u_h^n$ $(n = 0, 1, \ldots)$ which yield the convergence of numerical solutions. Thus we have

Theorem 3 (Basic estimates). *Assume $u_h^0 \in V_h$. Then u_h^n either becomes extinct or belongs to V_h for each $n \geqslant 0$, and the following estimates hold for all $n \geqslant 0$:*

$$T_h^* \leqslant t_n + \frac{\|u_h^n\|_\infty}{c'}, \tag{35}$$

$$\ell_0 - a\|(u_h^0)_x\|_\infty t_n \leqslant \ell_n \leqslant r_n \leqslant r_0 + a\|(u_h^0)_x\|_\infty t_n, \quad \textit{if} \ u_h^{n+1} \not\equiv 0, \tag{36}$$

$$0 \leqslant u_h^n(x) \leqslant \max(\|u_h^0\|_\infty - c' t_n, 0) \quad \textit{on } \mathbb{R}^1, \tag{37}$$

$$\|(u_h^n)_x\|_\infty \leqslant \|(u_h^0)_x\|_\infty, \tag{38}$$

$$TV((u_h^n)_x) \leqslant TV((u_h^0)_x), \tag{39}$$

$$\|(u_h^{n+1} - u_h^n)/k_{n+1}\|_{L^1(\mathbb{R}^1)} \leqslant (m + a)\|u_h^0\|_\infty TV((u_h^0)_x) + c'(r_0 - \ell_0 + 2a\|(u_h^0)_x\|_\infty t_n), \tag{40}$$

$$\inf_{i\in\mathbb{Z}} \delta^2 u_i^0 \leqslant \inf_{i\in\mathbb{Z}} \delta^2 u_i^n. \tag{41}$$

Proof. When $c = 0$, it follows from [9, Lemmas 3.1, 3.2 and 5.1] that the inequalities (36)–(41) hold. For $u_h' = D_{h,k} u_h$ it can be easily shown by simple calculations that

$$\ell(u_h) \leqslant \ell(u_h') \leqslant r(u_h') \leqslant r(u_h), \qquad \|u_h'\|_\infty \leqslant \max(\|u_h\|_\infty - c'k, 0),$$

$$\|((u_h'))_x\|_\infty \leqslant \|(u_h)_x\|_\infty, \qquad TV((u_h')_x) \leqslant TV((u_h)_x),$$

$$\|(u_h' - u_h)/k\|_{L^1(\mathbb{R}^1)} \leqslant c'\{r(u_h) - \ell(u_h)\}, \qquad \inf_{i\in\mathbb{Z}} \delta^2 u_i \leqslant \inf_{i\in\mathbb{Z}} \delta^2 u_i'.$$

Hence (36)–(41) hold for $c > 0$ by these inequalities.

Since u_h^m also becomes the numerical solution with the initial value u_h^n for arbitrary integers m and n $(n^* + 1 \geqslant m > n \geqslant 0)$, we have from (37)

$$\|u_h^m\|_\infty \leqslant \max\{\|u_h^n\|_\infty - c'(t_m - t_n),\ 0\} \quad \text{for all } m > n.$$

Then

$$\|u_h^m\|_\infty = 0 \quad \text{for } t_m \geqslant t_n + \frac{\|u_h^n\|_\infty}{c'},$$

which gives (35), and the proof is complete. □

Theorem 4 (Convergence of numerical solutions). *Assume $v^0(x)$ satisfies Conditions* (A-2) *and* (W-1) *with $\varphi = v^0$. Let $\{h\}$ be an arbitrary sequence which tends to zero. Then, there exists the unique weak solution v of* (1)–(2) *satisfying*

$$\|v_h - v\|_{L^\infty(\mathbf{H})} \to 0 \quad \textit{and} \quad |T_h^* - T^*| \to 0 \quad \textit{as} \quad h \to 0, \tag{42}$$

$$\operatorname{supp} v \subset [0,\ T^*] \times [\ell_0 - aC_1 T^*,\ r_0 + aC_1 T^*], \tag{43}$$

where $\mathbf{H}=[0,\infty)\times\mathbb{R}^1$, $v_h\equiv(u_h)^{1/(m-1)}$, $u_h(t,x)=u_h^n(x)$ *on* $[t_n,t_{n+1})\times\mathbb{R}^1$ *for all* t_n *and* h, *and* T^* *is the extinction time.*

Proof. Following Graveleau and Jamet's argument used in the proof of [4, Lemma 6.1 and Theorem 7.1], we can show that there exist a subsequence $\{h'\}$ of $\{h\}$ and a function U with the following properties:

(i) $U\in C^0(\mathbf{H})\cap L^\infty(\mathbf{H})$, $U_x\in L^\infty(\mathbf{H})$;
(ii) $U(0,x)=u^0(x)$ for all $x\in\mathbb{R}^1$;
(iii) $U(t,x)$ has compact support in $\mathbf{H}$;
(iv) As $h'\to 0$, $\|u_h'-U\|_{L^\infty(\mathbf{H})}$ and $\|(u_h')_x-U_x\|_{L^p(\mathbf{H})}\to 0$, where $1\leqslant p<\infty$;
(v) U becomes a weak solution of (15)–(16); that is, U satisfies

$$\iint_{\mathrm{supp}\,U}(U\phi_t-mUU_x\phi_x-(m-a)(U_x)^2\phi-c'\phi)\,\mathrm{d}x\,\mathrm{d}t+\int_{\mathbb{R}^1}U(0,x)\phi(0,x)\,\mathrm{d}x=0$$

for any function $\phi(t,x)\in C^{1,1}(\mathbf{H})$ with compact support.

Here the estimates (37)–(40) play an important role in proving these properties. By putting $v=U^{(1/m-1)}$, U becomes the unique weak solution of (1)–(2), which implies that $\|v_h-v\|_{L^\infty(\mathbf{H})}\to 0$ as $h\to 0$ and that (43) holds by (36). Moreover, it follows from (35) that $|T_h^*-T^*|\to 0$ as $h\to 0$, which completes the proof. □

4. Proofs of Theorems 1 and 2

4.1. Proof of Theorem 1

Let $\{u_h^n\}_{n=0,1,2,\dots}$ be the difference approximations given by (17) with $u^0(x)=(v^0(x))^{m-1}$, and put

$$T'=\tilde{T}+\tfrac{1}{2}\left\{\min(T_1,\ T_2)-\tilde{T}\right\},\tag{44}$$

where

$$\tilde{T}=\frac{\|u^0\|_{L^1(\gamma_1,\gamma_2)}}{c'(\gamma_2-\gamma_1)-(m+a)C_0TV(u_x^0)}\quad\text{and}\quad T_j=\frac{u^0(\beta_j)}{c'+mC_0C_2}\quad(j=1,2).$$

Suppose the solution $v(t,x)$ is positive on $\mathbf{S}=[0,\ T']\times[\gamma_1,\gamma_2]$. From (26), (27) and Theorem 4, there exists a positive number $h'(\eta)$ for the constant $\eta=\frac{1}{2}\min_{\mathbf{S}}u(t,x)>0$ such that

$$u_h^n(x)-c'k_{n+1}\geqslant u_h(t_n,x)-2\|u_x^0\|_\infty h>\eta\quad\text{for }(t_{n+1},x)\in\mathbf{S}\text{ and }h<h'.$$

Using the definitions of $D_{h,k}$, $H_{h,\tau}$ and $P_{h,\kappa}$ and Theorem 3, we have

$$\|D_{h,k_{n+1}}u_h^n\|_{L^1[\gamma_1,\ \gamma_2]} = \|u_h^n\|_{L^1[\gamma_1,\ \gamma_2]} - c'(\gamma_2-\gamma_1)k_{n+1},$$

$$\left\|\left(\prod_{j=1}^{v} H_{h,\tau_j}\right) D_{h,k_{n+1}}u_h^n - D_{h,k_{n+1}}u_h^n\right\|_{L^1[\gamma_1,\ \gamma_2]} \leqslant aC_0 TV(u_x^0)k_{n+1},$$

$$\left\|u_h^{n+1} - \left(\prod_{j=1}^{v} H_{h,\tau_j}\right) D_{h,k_{n+1}}u_h^n\right\|_{L^1[\gamma_1,\ \gamma_2]} \leqslant mC_0 TV(u_x^0)k_{n+1},$$

which yield

$$\|u_h^{n+1}\|_{L^1[\gamma_1,\ \gamma_2]} \leqslant \|u_h^0\|_{L^1[\gamma_1,\ \gamma_2]} - \{c'(\gamma_2-\gamma_1)-(m+a)C_0TV(u_x^0)\}t_{n+1}$$
$$\text{for } t_{n+1}\in[0,\ T'] \text{ and } h<h'. \tag{45}$$

Hence

$$\begin{aligned}\|u_h^{n+1}\|_{L^1[\gamma_1,\ \gamma_2]} &\leqslant \|u_h^0\|_{L^1[\gamma_1,\ \gamma_2]} - \|u^0\|_{L^1[\gamma_1,\ \gamma_2]} \\ &\quad -\{c'(\gamma_2-\gamma_1)-(m+a)C_0TV(u_x^0)\}(T'-\tilde{T}) \\ &< 0 \text{ for } t_{n+1}\in(\tilde{T},\ T'] \text{ and for sufficiently small } h,\end{aligned} \tag{46}$$

which is a contradiction. Thus

$$u(\tilde{t},\ \tilde{x}) = 0 \quad \text{for some} \quad (\tilde{t},\tilde{x})\in\mathbf{S}. \tag{47}$$

On the other hand, it follows that

$$(D_{h,k_{n+1}}u_h^n)(\beta_j) \geqslant u_h^n(\beta_j) - c'k_{n+1},$$

$$\left(\left(\prod_{j=1}^{v} H_{h,\tau_j}\right) D_{h,k_{n+1}}u_h^n - D_{h,k_{n+1}}u_h^n\right)(\beta_j) \geqslant 0,$$

$$\left(u_h^{n+1} - \left(\prod_{j=1}^{v} H_{h,\tau_j}\right) D_{h,k_{n+1}}u_h^n\right)(\beta_j) \geqslant -mC_0C_2k_{n+1},$$

which yield

$$u_h^{n+1}(\beta_j) \geqslant u_h^0(\beta_j) - (c'+mC_0C_2)t_{n+1} \geqslant u_h^0(\beta_j) - u^0(\beta_j) + (c'+mC_0C_2)(T_j-T')$$

for $t_{n+1}\in[0,\ T']$ $(j=1,2)$. Letting $h\to 0$, we have

$$u(t,\ \beta_j) \geqslant (c'+mC_0C_2)(T_j-T') > 0 \quad \text{for } t\in[0,\ T'] \ (j=1,2). \tag{48}$$

Hence, the first assertion of the theorem holds by (47) and (48). The second assertion follows from the fact that the numerical support monotonously shrinks as time increases under (7), which completes the proof. □

4.2. Proof of Theorem 2

We first note that $\tilde{K}(t,x,\rho,\sigma) \equiv (K(t,x,\rho,\sigma))^{m-1}$ satisfies

$$\tilde{K}(t,0,\rho,\sigma) = \tilde{K}(0,0,\rho,\sigma) - f(t,\rho,\sigma) - c't \quad \text{for } t \leqslant T^*(\rho,\sigma). \tag{49}$$

Now we prove the theorem. From the assumption of the theorem $u(t,x) = (v(t,x))^{m-1}$ becomes an even function and attains a local maximum or a local minimum at $x = 0$ for all later times $t > 0$. By [3, Proposition 2.4] the number of local maximum points of $u(t,x)$ is nonincreasing for $t > 0$. Therefore, to prove the theorem it suffices to show that $u(t^*,x)$ attains just *one* local maximum at $x = 0$ for some finite time $t^*(< T^*(\rho,\sigma_2))$. Assume the contrary; that is, suppose $u(t,x)$ has attains a local minimum at $x = 0$ for all $t < T^*(\rho,\sigma_2)$. Then we have

$$u_x(t,0) = 0 \quad \text{and} \quad u_{xx}(t,0) \geqslant 0 \quad \text{for all } t < T^*(\rho,\sigma_2). \tag{50}$$

From (15)–(16) and (13) it follows that

$$\begin{aligned} u(t,0) &= u^0(0) + \int_0^t \{mu(t,0)u_{xx}(t,0) + a(u_x(t,0))^2 - c'\}\,\mathrm{d}t \\ &\geqslant u^0(0) - c't \geqslant \tilde{K}(0,0,\rho,\sigma_2) - c't \quad \text{for all } t < T^*(\rho,\sigma_2). \end{aligned} \tag{51}$$

Since $u(t,0) \leqslant \tilde{K}(t,0,\rho,\sigma_1)$ holds by (13) and the comparison theorem which is concerned with the initial data [2] we have from (49) and (51)

$$\tilde{K}(0,0,\rho,\sigma_1) - \tilde{K}(0,0,\rho,\sigma_2) \geqslant f(t,\rho,\sigma_1) \quad \text{for all } t < T^*(\rho,\sigma_2), \tag{52}$$

which yields

$$\frac{\rho^2}{m-1}\left(\frac{1}{\sigma_1} - \frac{1}{\sigma_2}\right) \geqslant f(T^*(\rho,\sigma_2),\rho,\sigma_1). \tag{53}$$

This inequality contradicts to (14). Hence there exists some positive constant $\tilde{t}(< T^*(\rho,\sigma_2))$ such that $u(\tilde{t},x)$ attains a local maximum at $x = 0$; that is,

$$u_x(\tilde{t},0) = 0 \quad \text{and} \quad u_{xx}(\tilde{t},0) < 0. \tag{54}$$

Thus the number of local maximum points of $u(\tilde{t},x)$ is equal to *one*. We therefore merely put $t^* = \tilde{t}$ to complete the proof. □

5. Numerical examples

In this section we show some numerical computations. Fig. 1 shows the convergence rate of numerical approximations which are starting from Kersner's explicit solution at $t=0$ [9]. Our scheme gives good approximations.

We now show the numerical examples for Theorem 1. Figs. 2 and 3 show the numerical profiles of v_h and $\operatorname{supp} v_h$. The *support splitting phenomena* occur in Fig. 3, where the inequality (6) is satisfied. On the other hand, the inequality (6) with $u^0(x)=u_h^n$ $(n=0,1,2,\dots)$ is not satisfied in Fig. 2.

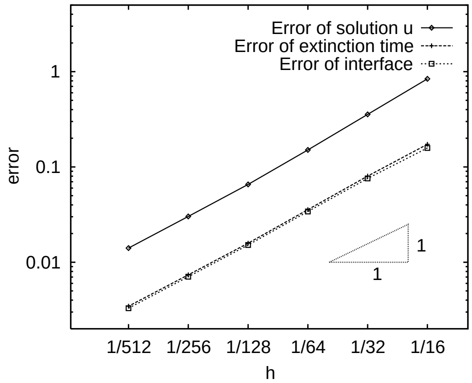

Fig. 1. Relative errors versus space mesh width h with $m = 1.5$, $p = 0.5$ and $c = 1$.

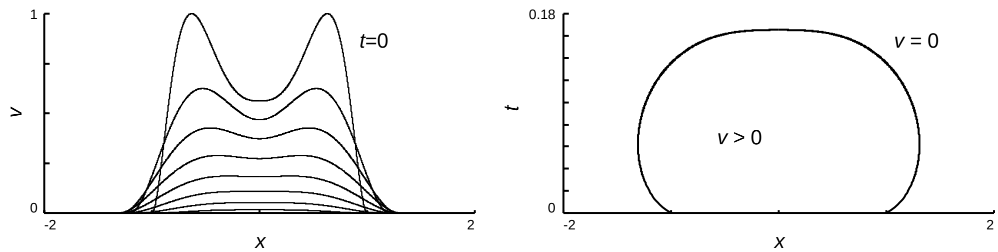

Fig. 2. Numerical solutions and interface with $m = 1.5$, $p = 0.5$, $c = 10$ and $h = \frac{1}{1024}$.

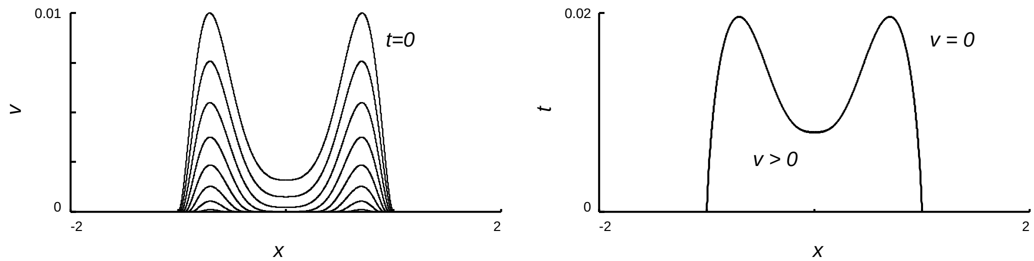

Fig. 3. Numerical solutions and interface with $m = 1.5$, $p = 0.5$, $c = 10$ and $h = \frac{1}{1024}$.

Fig. 4 shows *the numerical prediction time* t_p by which the support splits for the initial function $v_d^0(x)$, where

$$t_p = t + \frac{\|u_h(t,\cdot)\|_{L^1(\gamma_1,\gamma_2)}}{c'(\gamma_2 - \gamma_1) - (m+a)\|u_h(t,\cdot)\|_\infty TV((u_h)_x(t,\cdot))}, \quad v_d^0 = (u_d^0(x))^{\frac{1}{m-1}}$$

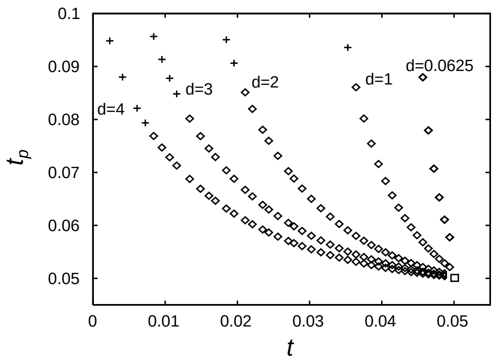

Fig. 4. Numerical predictions t_p at each time t with $m = 1.5$, $p = 0.5$, $c = 10$ and $h = \frac{1}{256}$.

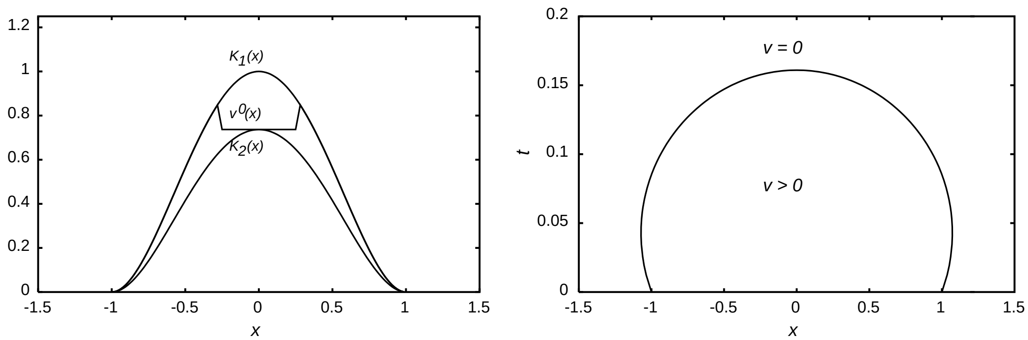

Fig. 5. Initial functions and numerical interface with $m = 1.5$. $p = 0.5$, $c = 10$ and $h = \frac{1}{64}$.

and

$$u_d^0(x) = u_d^0(-x) = \begin{cases} 0.25 & \text{if } 0 \leqslant x \leqslant d, \\ 0.25 \times 2^{-\frac{|x-d-1|^3}{1-|x-d-1|^3}} + 0.25 & \text{if } d < x \leqslant d+1, \\ -0.5(x-d-1)^3 + 0.5 & \text{if } d+1 < x < d+2, \\ 0 & \text{if } d+2 \leqslant x. \end{cases} \tag{55}$$

When the inequality (6) holds with $u^0(x) = u_h(t,x)$, the point (t, t_p) is denoted by $\diamondsuit$. Otherwise, (t, t_p) is denoted by $+$. The time at which the numerical support splits into two disjoint sets is denoted by $\square$. The criterion (6) for (C-1) gives good numerical predictions.

Next we show the numerical examples for Theorem 2. Figs. 5–7 show the numerical profiles of the initial functions $v^0(x)$, $K_1(x)$ and $K_2(x)$ and numerical interfaces, where $K_1(x) = K(0,x,1,2)$, $K_2(x) = K(0,x,1,2.33)$ and the inequality (14) holds with $\rho = 1$, $\sigma_1 = 2$ and $\sigma_2 = 2.33$. In Fig. 5 the inequality $K_2(x) \leqslant v^0(x) \leqslant K_1(x)$ (see (13)) holds and the support never splits by Theorem 2. We try numerical computations to the initial functions for which the inequality $K_2(x) \leqslant v^0(x) \leqslant K_1(x)$ fails. The *support nonsplitting phenomena* continue for the small perturbation which is imposed on

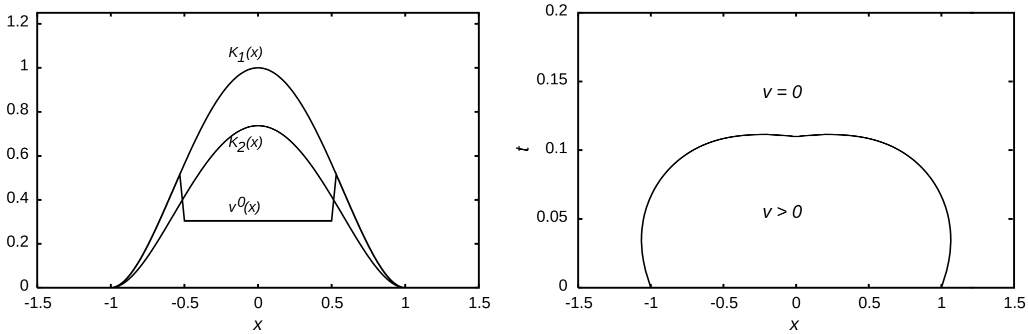

Fig. 6. Initial functions and numerical interface with $m = 1.5$, $p = 0.5$, $c = 10$ and $h = \frac{1}{64}$.

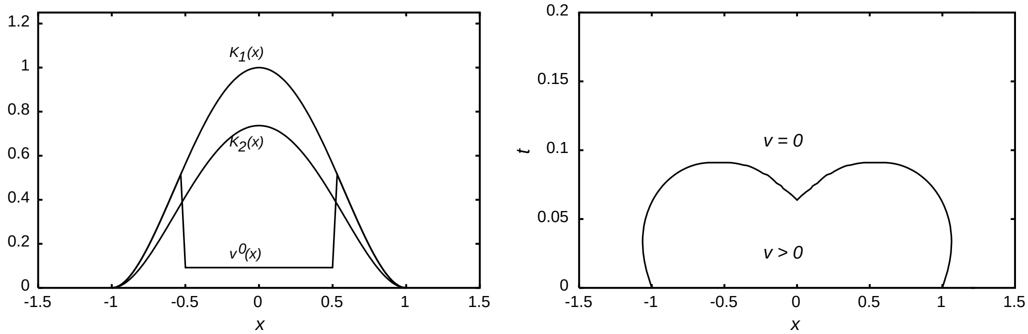

Fig. 7. Initial functions and numerical interface with $m = 1.5$, $p = 0.5$, $c = 10$ and $h = \frac{1}{64}$.

$v^0(x)$ in Fig. 5. In Fig. 6 the *support splitting phenomena* begin to appear. From these numerical results the improvement of our criterion (13)–(14) for (C-2) is needed.

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